Generating Functions for Solving Recurrence Relations

Diving In

The general approach here is a bit cumbersome to explain out of context so let's dive in and do it by example. We'll take the recurrence relation for the Fibonacci numbers as our example as it is notoriously difficult to do by back-substitution.

The Problem

Our goal here is to solve the Fibonacci sequence into closed form with generating functions. Let's start with the recurrence relation for the sequence:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2}, n > 1 \end{aligned}$$

Now, for our generating function. Our goal will be a generating function that holds all the Fibonacci numbers as coefficients:

$$F(z) = \sum_{n \geq 0} f_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \ldots$$

The Process Begins

To massage our recurrence equation to look like the initial equation form, we'll start by multiplying our recurrence by our z^n powers:

$$f_n z^n = f_{n-1} z^n + f_{n-2} z^n, n > 1$$

This puts those Fibonacci numbers in as coefficients of the z^n powers — just not the right ones, yet.

Next we'll sum up those equations over all valid n values. The one we've written represents a whole sequence of equations, after all. Remember from back-substitution that we use that form/template to make the substitution equations for f_{n-1}, f_{n-2}, etc.

Also remember from your solution of systems of equations in algebra that we can add two equations together to get a new equation that is still valid. Here we are taking that to a bit of an extreme. The result:

$$\sum_{n>1} f_n z^n = \sum_{n>1} f_{n-1} z^n + \sum_{n>1} f_{n-2} z^n$$

This notation with no top limit on the summation just means we are summing to infinity — an infinitely long sum starting where we specify below. Also note that n is an integer here so $n > 1$ means n starts at 2.

Now, let's make the left side look like $F(z)$ by including the missing terms. Particularly f_0 and f_1 are missing. To add things to a single side of an equation is illegal, of course, unless you both add them and subtract them at the same time! So we do that here:

$$-1 - 1z + \left(1 + 1z \sum_{n>1} f_n z^n\right) = -1 - 1z + \sum_{n\geq 0} f_n z^n = F(z) - z - 1$$

The other two sums will require a little shifting to line up with the powers and subscripts. To do that, we'll factor out some zs. We can do this because z is not related to the index of summation. (It's exponent is, but it is considered constant with respect to the index!)

$$F(z) - z - 1 = z \sum_{n>1} f_{n-1} z^{n-1} + z^2 \sum_{n>1} f_{n-2} z^{n-2}$$

Now we can substitute $k = n - 1$ and $m = n - 2$ into the two right-side sums to simplify them:

$$F(z) - z - 1 = z \sum_{k>0} f_k z^k + z^2 \sum_{m\geq 0} f_m z^m$$

Note that to find the new lower bounds on the sums we use the old bounds in the substitution equations. So, for instance, when $n = 1$, $k = 1 - 1 = 0$. And when $n = 1$, $m = 1 - 2 = -1$ and so it is $\geq$ zero — again we are all integer indices here!

Now the m summation is ready to be $F(z)$ — the index names don't matter! — and the k one is just a single term off, so we add and subtract that f_0 term to the left sum. Note that this has to happen inside parentheses so that the z we pulled out earlier affects the new terms, too!

$$\begin{aligned}
F(z) - z - 1 &= z\left(\sum_{k>0} f_k z^k + 1 - 1\right) + z^2 F(z) \\
&= z\left(\left(\sum_{k>0} f_k z^k + 1\right) - 1\right) + z^2 F(z) \\
&= z\left(\sum_{k\geq 0} f_k z^k - 1\right) + z^2 F(z) \\
&= z(F(z) - 1) + z^2 F(z)
\end{aligned}$$

Next we collect the $F(z)$ terms to the left side and everything else to the right and massage it a bit to solve for $F(z)$:

$$\begin{aligned}
F(z) - zF(z) - z^2 F(z) &= z - z + 1 \\
F(z)(1 - z - z^2) &= 1 \\
F(z) &= \frac{1}{1 - z - z^2}
\end{aligned}$$

And this should be it, right? But that's just the generating function! We don't have the closed form, yet.

The Closed Form

To get the closed form, we must simplify that rational expression. We have a crazy denominator here. It looks like:

$$1 - z - z^2 = (1 + az)(1 + bz)$$

But for what a and b? Well, it turns out that they are:

$$\frac{-1 \pm \sqrt{5}}{2}$$

One way to solve this is partial fraction decomposition.[1] Remember that we take the factors of the polynomial and put each as a separate rational denominator and add them together. As their numerators, we invent new variables to be discovered via a system of equations. (Since our factors are linear, we just have single constants — not small polynomials in their own right.)

So here we have:

$$\begin{aligned}
\frac{1}{1-z-z^2} &= \frac{A}{1+az} + \frac{B}{1+bz} \qquad = A\sum_{n\geq 0}(-a)^n z^n + B\sum_{n\geq 0}(-b)^n z^n \\
&= \frac{A(1+bz)+B(1+az)}{(1+az)(1+bz)}
\end{aligned}$$

(That extra = there reminds us what the generating functions look like in summation form. This will play a role later!)

Next we focus on the numerators by setting them equal:

$$\begin{aligned}
1+0z &= A+Abz+B+Baz \\
&= A+B+(Ab+Ba)z
\end{aligned}$$

So, it would seem that $A+B=1$ so we know, for instance, that $B=1-A$.

From the coefficient of z we also know that:

$$\begin{aligned}
0 &= Ab+Ba \\
0 &= Ab+(1-A)a \\
-a &= A(b-a) \\
A &= \frac{a}{a-b} \\
&= \frac{5-\sqrt{5}}{10}
\end{aligned}$$

So, B must be:

$$\begin{aligned}
B &= 1-\frac{a}{a-b} \\
&= \frac{a-b-a}{a-b} \\
&= \frac{b}{b-a} \\
&= \frac{5+\sqrt{5}}{10}
\end{aligned}$$

Thus our solutions are:

$$A(-a)^n + B(-b)^n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

[1] Don't worry about the general practice here. I promise not to make these harder than this one. If you want to see more, try looking for this topic at wikipedia or PurpleMath.

(Remember, just the coefficients are our solutions! The rest of the summation notation is just the clothesline we hang them on.)

Exercises

1. Find the closed form solution of $a_0 = 3$ and $a_n = 5a_{n-1} + 1$ when $n \geq 1$. $*$
2. Find the closed form solution of $a_0 = 1$, $a_1 = 8$, and $a_n = 7a_{n-1} - 12a_{n-2}$ when $n \geq 2$.

Solutions to Starred Exercises

1. Find the closed form solution of $a_0 = 3$ and $a_n = 5a_{n-1} + 1$ when $n \geq 1$.
   Our goal is a generating function like $A(z) = \sum_{n\geq 0} a_n z^n$. The recurrence equation is first multiplied through by z^n:
   $$a_n z^n = 5a_{n-1}z^n + z^n, n > 0$$
   Then these equations are added up over all legitimate n values:
   $$\sum_{n>0} a_n z^n = 5\sum_{n>0} a_{n-1}z^n + \sum_{n>0} z^n$$
   Now we simplify by adding/subtracting terms from the sums to make sums we know:
   $$\sum_{n>0} a_n z^n + a_0 - a_0 = 5\sum_{n>0} a_{n-1}z^n + \sum_{n>0} z^n + 1 - 1$$
   Renaming these, we can factor zs to make the remaining sum closer:
   $$A(z) - 3 = 5z\sum_{n>0} a_{n-1}z^{n-1} + \frac{1}{1-z} - 1$$
   Then a quick change of index ($k = n - 1$):
   $$A(z) - 3 = 5zA(z) + \frac{1}{1-z} - 1$$
   Collect the $A(z)$s to one side and the rest to the other:
   $$\begin{aligned} A(z) - 5zA(z) &= \tfrac{1}{1-z} + 2 \\ A(z)(1-5z) &= \tfrac{1}{1-z} + 2 \\ A(z) &= \tfrac{1}{(1-z)(1-5z)} + \tfrac{2}{1-5z} \end{aligned}$$
   This would be:
   $$A(z) = \left(\sum_{n\geq 0} z^n\right)\left(\sum_{n\geq 0} 5^n z^n\right) + 2\sum_{n\geq 0} 5^n z^n$$
   And here we can use the product rule from our MIT notes to simplify the left sum:
   $$A(z) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} 1 \cdot 5^{n-k}\right) z^n + 2\sum_{n\geq 0} 5^n z^n$$

Luckily 5^{n-k} as k runs from 0 to n is the same added together as 5^k would be so we simplify:

$$A(z) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} 5^k\right) z^n + 2\sum_{n\geq 0} 5^n z^n$$

And this is a partial sum of a geometric sequence we learned in the summation section so long ago (I'll also bring in the 2 from in front of the other sum for convenience):

$$\begin{aligned}
A(z) &= \sum_{n\geq 0}\frac{5^{n+1}-1}{5-1}z^n + \sum_{n\geq 0} 2\cdot 5^n z^n \\
&= \sum_{n\geq 0}\left(\frac{1}{4}(5^{n+1}-1) + 2\cdot 5^n\right) z^n \\
&= \frac{1}{4}\sum_{n\geq 0}(5^{n+1} + 8\cdot 5^n - 1)z^n \\
&= \frac{1}{4}\sum_{n\geq 0}(5^n(5+8) - 1)z^n \\
&= \frac{1}{4}\sum_{n\geq 0}(13\cdot 5^n - 1)z^n
\end{aligned}$$

So our solutions are $a_n = \dfrac{13}{4}5^n - \dfrac{1}{4}$ for $n \geq 0$.